

Thursday 27 March

HW clarification

$$3) \phi = \frac{p+v}{\sqrt{2}} e^{i\theta} \rightarrow \frac{p+v}{\sqrt{2}} e^{i\theta/v}$$

b) you can either add  $(\partial_u \phi)^4$  or  $\left(\frac{\partial_u \theta}{v}\right)^4$

$$c) C_4^r(\mu, \varepsilon)$$

remove infinities,  $\frac{1}{\varepsilon} C_4(\mu, \varepsilon) \rightarrow C_4^r(\mu)$

d) determine  $\mu$  dependence of  $C_4^r(\mu)$

e.g.

$$\delta A(s) \approx \frac{1}{2} \left( \frac{s}{4\pi v^2} \right)^2 \left( -\ln \frac{m^2}{\mu^2} \right)$$

$$\text{suppose } C_4^r(\mu) \frac{(\partial_u \theta)^4}{4v} \frac{1}{(4\pi)^2}$$

$$\Rightarrow \delta A(s) \approx \frac{s^2}{(4\pi v^2)^2} C_4^r(\mu)$$

$$= \delta \bar{A}(s) = \left( \frac{s}{4\pi v^2} \right)^2 \left[ -\frac{1}{2} \ln \frac{m^2}{\mu^2} + C_4^r(\mu) \right]$$

$$\text{We demand } \frac{\partial}{\partial \ln \mu^2} \delta \bar{A}(s) = 0$$

$$\Rightarrow \frac{1}{2} + \frac{\partial}{\partial \ln \mu^2} C_4^r(\mu) = 0$$

$$\Rightarrow C_4^r(\mu) = \frac{1}{2} \ln \left( \frac{\mu^2}{\mu_0^2} \right) + C_4^r(\mu_0)$$

$$\Rightarrow \delta \bar{A}(s) = \left( \frac{s}{4\pi v^2} \right)^2 \left[ -\frac{1}{2} \ln \frac{m^2}{\mu^2} + C_4^r(\mu_0) - \frac{1}{2} \ln \frac{m^2}{\mu_0^2} \right]$$

Also, for  $\pi \rightarrow l\bar{l}$

$$\bar{u} \gamma_\mu (1-\gamma_5) v$$

$$r \propto |A|^2$$

$$(\bar{u} \gamma_\mu (1-\gamma_5) v)^+$$

$$= v^+ (1-\gamma_5)^+ \gamma_\mu^+ (\bar{u})^+$$

$$= v^+ \gamma_0 \gamma_0 (1-\gamma_5) \gamma_\mu^+ \gamma_0^+ v$$

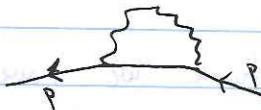
$$= \bar{v}^+ (1+\gamma_5) \gamma_0 \gamma_\mu^+ \gamma_0 v$$

$$= \bar{v} (1+\gamma_5) \gamma_\mu v$$

## Electromagnetic Self-Energy of the nucleon:

~~Using a cut Cottingham Formula~~

We know how to compute the leading QED correction to the self energy of the  $e^-$



$$-i \Sigma(p) = (ie)^2 \frac{1}{2} \sum_{\sigma} \int_R \frac{d^4 k}{(2\pi)^4} \bar{u}(p) \gamma_\mu \frac{g_{\mu\nu}(-i)}{k^2 + i\epsilon} \frac{i}{k - k_0 - m + i\epsilon} \gamma_\nu u(p)$$

$$\delta m = \frac{\alpha m}{4\pi} \left[ \frac{3}{2} - 3 \ln \frac{m^2}{\Lambda^2} \right] + \delta m_{\text{ct.}}(\Lambda)$$

aside: the coefficient of the  $\ln m^2$  is universal - independent of regularization prescription.

Why?

The  $\ln m^2$  contribution can not be absorbed by a finite number of local operators.

It is an infrared-long-distance effect

so-momentum cutoff

Pauli-Villars

dim-reg

:

all must have the same coefficient of the  $\ln m^2$  term

We know the proton can not be approximated by a point fermion

$$\bar{u}(p) \Gamma_\mu u(p) = \bar{u}(p) \left[ \gamma_\mu F_1(Q^2) + i \frac{\sigma^\nu Q^\mu}{2M} F_2(Q^2) \right] u(p)$$

for on-shell proton matrix elements

not to mention the inelastic structure we have been discussing.

- But, can we use this info to determine the electromagnetic self-energy of the proton?

We have known the neutron is slightly heavier than the proton since nearly we knew of it's existence.

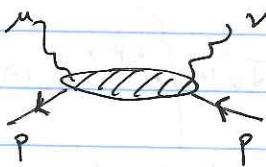
Before QCD, the only isospin violation we knew of was QED. People worked hard to see if EM could make the neutron heavier than the proton.

We now understand the answer comes also from  $m_d - m_u$ .

We also know how to evaluate the EM self-energy.

Begin with the forward Compton Scattering Amplitude

$$T_{\mu\nu} = \frac{i}{2} \sum \int d^4 z e^{iqz} \langle p\sigma | T \{ J_\mu(z) J_\nu(0) \} | p\sigma \rangle$$



As with the hadronic tensor,  $W_{\mu\nu}$ , there are only 2 Lorentz structures we can make

$$T_{\mu\nu} = -D_{\mu\nu}^{(1)} T_1(v, -q^2) + D_{\mu\nu}^{(2)} T_2(v, -q^2)$$

$$= d_{\mu\nu}^{(1)} q^2 t_1(v, -q^2) \mp d_{\mu\nu}^{(2)} q^2 t_2(v, -q^2)$$

$$D_{\mu\nu}^{(1)} = d_{\mu\nu}^{(1)} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}$$

$$d_{\mu\nu}^{(2)} = \frac{1}{q^2} \left[ P_\mu P_\nu - \frac{P \cdot q}{q^2} (P_\mu q_\nu + P_\nu q_\mu) + \frac{(P \cdot q)^2}{q^2} g_{\mu\nu} \right]$$

$$D_{\mu\nu}^{(2)} = \frac{1}{q^2} \left( P_\mu - q_\mu \frac{P \cdot q}{q^2} \right) \left( P_\nu - q_\nu \frac{P \cdot q}{q^2} \right)$$

These are just 2 of many possible choices for the decomposition of  $T_{\mu\nu}$ .

190

What do we know about  $T_{\mu\nu}$ ?

$$T_{\mu\nu} = \frac{i}{2} \sum_{\sigma} \int d^4 z e^{i q \cdot z} \langle p\sigma | T \{ J_\mu(z) J_\nu(0) \} | p\sigma \rangle$$

$$= \frac{i}{2} \sum_{\sigma} \int_{\Gamma} d^4 z e^{i q \cdot z} \left[ \Theta(z^0) \langle p\sigma | e^{i \hat{p} \cdot z} J_\mu(0) e^{-i \hat{p} \cdot z} J_\nu(0) | p\sigma \rangle + \Theta(-z^0) \langle p\sigma | J_\nu(0) \left[ e^{i \hat{p} \cdot z} J_\mu(0) e^{-i \hat{p} \cdot z} | p\sigma \rangle \right] \right]$$

 $|\Gamma X \Gamma|$ 

$$= \frac{i}{2} \sum_{\sigma} \int_{\Gamma} d^4 z \left[ \Theta(z^0) e^{i q \cdot (q + p - p_f)} \langle p\sigma | J_\mu |\Gamma X \Gamma | J_\nu | p\sigma \rangle + \Theta(-z^0) e^{i q \cdot (q - p + p_f)} \langle p\sigma | J_\nu |\Gamma X \Gamma | J_\mu | p\sigma \rangle \right]$$

We can now perform  $\int d^4 z$  integral

$$\int d^4 z \Theta(z^0) e^{i z^0 (q^0 - (p_f^0 - p_0))}$$

$$= -i \frac{e^{i z^0 (q^0 - (p_f^0 - p_0)) + i\varepsilon}}{q^0 - (p_f^0 - p_0 - i\varepsilon)} \Big|_0^\infty$$

The  $+i\varepsilon$  is needed to make sense of the  $\infty$  limit. Adding  $+i\varepsilon$  leads to  $e^{-\infty\varepsilon}$ , damping this part of integral!

$$= \frac{-i}{q^0 - (p_f^0 - p_0 - i\varepsilon)}$$

Similarly

$$\int d^4 z \Theta(-z^0) e^{i z^0 (q^0 + p_f^0 - p^0)}$$

$$= \frac{-i}{q^0 + (p_f^0 - p^0 - i\varepsilon)}$$

$$T_{\text{ur}} = \frac{1}{2} \sum_{\sigma} \oint_{\Gamma} \left\{ - (2\pi)^3 S^3 (\vec{q} - \vec{p}_r + \vec{p}) \frac{\langle p\sigma | J_\mu | r \times r | J_\nu | p\sigma \rangle}{q^0 - (p_r^0 - p^0 - i\epsilon)} + (2\pi)^3 \delta^3 (\vec{q} + \vec{p}_r - \vec{p}) \frac{\langle p\sigma | J_\nu | r \times r | J_\mu | p\sigma \rangle}{q^0 + (p_r^0 - p^0 - i\epsilon)} \right\}$$

$$\oint_{\Gamma} = \sum_{\Gamma} \int \frac{d^3 p_r}{(2\pi)^3} \frac{1}{2E_p}$$

What are these states  $|r\rangle$  ?

$$\begin{array}{c} \left. \begin{array}{c} + \\ \leftarrow \end{array} \right\} \downarrow q \\ \leftarrow \left( \begin{array}{c} 0 \\ \leftarrow \end{array} \right) \leftarrow \end{array} \quad |r\rangle = |\text{proton, on shell}\rangle$$

$|\Delta\rangle$

Lets first focus on the on-shell proton state.

(This is like taking only the u-spinor term)

$$\psi(k) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} [a_k u_k e^{-ik \cdot x} + b_k^\dagger v_k e^{ik \cdot x}]$$

$$\langle p\sigma | J_\mu | p\sigma' \times p\sigma' | J_\nu | p\sigma \rangle$$

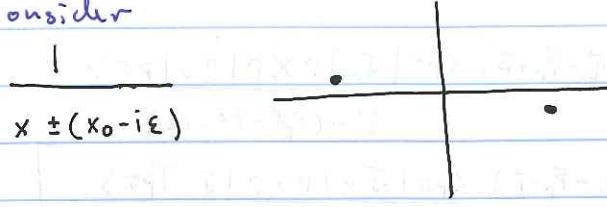
The three dimensional delta functions will ensure this proton state has the momentum

$$\vec{p}_r = \vec{p} + \vec{q}$$

But to make sense of this matrix element, we need to put the proton on its "mass-shell", meaning get the energy to satisfy  $p_r^2 = M^2$

To get this extra constraint, we can take the imaginary part of  $T_{\text{ur}}$

Consider



We can decompose this function into its real & imaginary part

$$\frac{1}{x \pm (x_0 - i\epsilon)} = P\left(\frac{1}{x \pm x_0}\right) + i\pi \delta(x \pm x_0)$$

↑                      ↑  
Principle Value      Imaginary Part

How do we see this?

$-x_0 + i\epsilon$

The  $+i\epsilon$  is the same as a slight contour deformation in the counter-clock wise direction by an angle  $\pi$ . We have "half" enclosed the pole so Cauchy  $\Rightarrow \frac{1}{2} \cdot 2\pi i \cdot \text{Residue}$

So, we see taking the imaginary part of  $T_{ur}$  gives us the energy conserving  $\vec{v}$  needed to put the intermediate proton "on mass-shell".

$$2\text{Im } T_{ur} = \frac{1}{2} \sum_{\sigma\sigma'} \int_F \left[ -(2\pi)^4 \delta^4(q - p_r + p) (-) \langle p\sigma | J_\mu | p_r\sigma' \times p_r\sigma' | J_\nu | p\sigma \rangle + (2\pi)^4 \delta^4(q + p_r - p) \langle p\sigma | J_\nu | p_r\sigma' \times p_r\sigma' | J_\mu | p\sigma \rangle \right]$$

What are these matrix elements?

$$\sum_{\sigma\sigma'} \langle p\sigma' | J_\mu | p_r\sigma' \times p_r\sigma' | J_\nu | p\sigma \rangle$$

$$= \sum_{\sigma\sigma'} \bar{u}_{\sigma'}(p) \left[ \gamma_\mu F_1(Q^2) + \frac{i\sigma_{\mu\nu}(p^\mu - p_r^\mu)}{2M} F_2(Q^2) \right] u_\sigma(p_r)$$

$$\times \bar{u}_{\sigma'}(p_r) \left[ \gamma_\nu \bar{F}_1(Q^2) + \frac{i\sigma_{\nu\rho}(p_r^\beta - p^\beta)}{2M} \bar{F}_2(Q^2) \right] u_\sigma(p)$$

Notice  $\sum_{\sigma\bar{\sigma}}$ ,  $\Rightarrow \text{Tr}\ \text{Lorentz indices} = < >$

Also, these spinors are now all on-shell, so we can use the Gordon Identity,

$$\bar{u}(p') \gamma_\mu u(p) = \bar{u}(p') \left[ \frac{p'_\mu + p_\mu}{2M} + i \frac{\gamma_{\mu\nu} (p'^\nu - p^\nu)}{2M} \right] u(p)$$

$\Rightarrow$

$$\sum_{\sigma\bar{\sigma}} \langle p\sigma | J_\mu | p_F \sigma' \times p_F \sigma' | J_\nu | p\tau \rangle$$

$$= \left\langle \bar{u}(p) \left[ \gamma_\mu F_1 + \bar{F}_2 \left( \gamma_\mu - \frac{p_{F\mu} + p_\mu}{2M} \right) \right] u(p_F) \bar{u}(p_F) \left[ \gamma_\nu F_1 + \bar{F}_2 \left( \gamma_\nu - \frac{p_{F\nu} + p_\nu}{2M} \right) \right] u_F(p) \right\rangle_{\text{Tr}}$$

$$p + M$$

$$= \left\langle (p+M) \left( \gamma_\mu (F_1 + \bar{F}_2) - \frac{(p_F + p)_\mu}{2M} \bar{F}_2 \right) (p_F + M) \left( \gamma_\nu (F_1 + \bar{F}_2) - \frac{(p_F + p)_\nu}{2M} \bar{F}_2 \right) \right\rangle$$

$$= \langle p \gamma_\mu p_F \gamma_\nu \rangle (F_1 + \bar{F}_2)^2 - \langle p \gamma_\mu \rangle (F_1 + \bar{F}_2) M \frac{p_{F\mu} + p_\mu}{2M} \bar{F}_2$$

$$- \langle p_F \gamma_\nu \rangle (F_1 + \bar{F}_2) M \frac{p_{F\mu} + p_\mu}{2M} \bar{F}_2$$

+ ...

$$+ M^2 \frac{p_{F\mu} + p_\mu}{2M} \frac{p_{F\nu} + p_\nu}{2M} \bar{F}_2^2 \langle 1 \rangle$$

$$+ M^2 \langle \gamma_\mu \gamma_\nu \rangle (F_1 + \bar{F}_2)^2$$

$$(p_F - p +) \cdot T = (p_F - p) \cdot T$$

$$\langle \not{P} \gamma_\mu \not{P} \gamma_\nu \rangle = 4(P_\mu P_\nu + P_\nu P_{\mu} - P \cdot P g_{\mu\nu})$$

$$\langle \not{P} \not{P} \rangle = 4 P \cdot P$$

$$P \cdot P = P \cdot (P \pm q)$$

$$\langle \not{P} \gamma_\mu \rangle = 4 P_\mu$$

$$= M^2 + M\nu \quad (q^0 = \nu)$$

$\leftarrow$  sign for  $\mu \leftrightarrow \nu$

$$\langle 1 \rangle = 4$$

$$P_P^2 = M^2 = (P+q)^2 = M^2 + 2P \cdot q + q^2$$

$$P_P = P + q$$

$$\Rightarrow 2M\nu = -q^2 \quad \begin{matrix} \text{on shell} \\ \text{only} \end{matrix}$$

$$\sum_{\sigma\sigma'} \langle p\sigma | J_\mu | p'_\mu \rangle \langle p'_\mu \sigma' | J_\nu | p\sigma \rangle$$

$$= -4M\nu (F_1 + F_2)^2 D_{\mu\nu}^{(1)} + 4(2M^2 F_1^2 + M\nu F_2^2) D_{\mu\nu}^{(2)}$$

$$- 2M\nu F_2 (F_1 + (2 - \frac{\nu}{M}) F_2) d_{\mu\nu}^{(1)} + 4(2M^2 F_1^2 + M\nu F_2^2) d_{\mu\nu}^{(2)}$$

What about 2nd term  $S^4(q + P_P - P)$

$$P_P = P - q$$

$$(P_P^2 = M^2 = M^2 - 2M\nu + q^2)$$

$$\Rightarrow 2M\nu = q^2$$

$$\sum_{\sigma\sigma'} \langle p\sigma | J_\nu | p'_\mu \sigma' | X_{P_P} \gamma^\mu | J_\mu | p\sigma \rangle, \quad \text{observe } D_{\mu\nu}^{(1)} = D_{\nu\mu}^{(1)} \\ d_{\mu\nu}^{(1)} = d_{\nu\mu}^{(1)}$$

So we see the only change compared to first term is  $\nu \rightarrow -\nu$

$$= +4M\nu (F_1 + F_2)^2 D_{\mu\nu}^{(1)} + 4(2M^2 F_1^2 - M\nu F_2^2) D_{\mu\nu}^{(2)}$$

This is "crossing symmetry"

$$T_{\mu\nu}(-\nu, -q^2) = T_{\mu\nu}(+\nu, -q^2)$$

Recall, we can relate these elastic form factors to the E,M form factors

$$F_1 + F_2 = G_M \quad F_1 = \frac{G_E - \frac{q^2}{4M^2} G_M}{1 - \frac{q^2}{4M^2}}$$

$$F_2 = \frac{G_M - G_E}{1 - \frac{q^2}{4M^2}}$$

And let's use the phase space integral

$$\int \frac{d^3 p_F}{(2\pi)^3} \frac{1}{2E_F} (2\pi)^4 \delta^4(p_F - (p+q))$$

$$= \int \frac{d^4 p_F}{(2\pi)^4} (2\pi) \delta(p_F^2 - M^2) (2\pi)^4 (p_F - (p+q))$$

$$= 2\pi \delta((p+q)^2 - M^2)$$

$$= 2\pi \delta(2Mv + q^2)$$

$$= 2\pi \cdot \frac{1}{2M} \delta(v + \frac{q^2}{2M})$$

Putting this all together, we get

$$2 \operatorname{Im} T_{uv}^{\text{e.l.}} = 2M \cdot 2\pi \cdot \delta(v + \frac{q^2}{2M}) \left\{ -D_{uv}^{(1)} \frac{-q^2}{4M^2} G_M^2(-q^2) + D_{uv}^{(2)} \frac{G_E^2(-q^2) - \frac{q^2}{4M^2} G_M^2(-q^2)}{1 - \frac{q^2}{4M^2}} \right. \\ \left. + d_{uv}^{(1)} \frac{-q^2}{4M^2} \frac{G_E^2 - G_M^2}{1 - \frac{q^2}{4M^2}} - d_{uv}^{(2)} \frac{-(G_E^2 - \frac{q^2}{4M^2} G_M^2) G_M^2}{1 - \frac{q^2}{4M^2}} \right\}$$

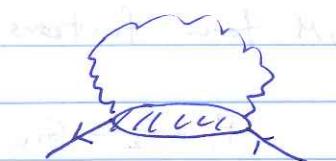
$$2 \operatorname{Im} T_1(v, q^2) = 2M \cdot 2\pi \cdot \delta(v + \frac{q^2}{2M}) \frac{-q^2}{4M^2} G_M^2(-q^2)$$

$$2 \operatorname{Im} T_2(v, q^2) = 2M \cdot 2\pi \cdot \delta(v + \frac{q^2}{2M}) \frac{G_E^2 - \frac{q^2}{4M^2} G_M^2}{1 - \frac{q^2}{4M^2}}$$

196

So what do we do with this?

$$i T_{\mu\nu} = \text{Diagram with two external lines } p^\mu \text{ and } p^\nu$$



What if we tie the photon propagator together and integrate over the 4-momentum?

$$\delta M = \frac{i}{2M} \frac{\alpha_{f.s.}}{(2\pi)^3} \int_R d^4q \frac{g^{\mu\nu} T_{\mu\nu}(p, q)}{q^2 + i\epsilon} + \delta M_R$$

$$\frac{1}{2M} : \langle p | q \rangle = 2E_p \delta^3(\vec{p} - \vec{q}) (2\pi)^3$$

$$\frac{\alpha_{f.s.}}{(2\pi)^3} = \frac{1}{2} \frac{e^2}{(2\pi)^4} \quad \frac{(2\pi)^4 - d^4q}{2} = 2^{\text{nd}} \text{ order perturbation theory}$$

R: the integral will have to be renormalized  
 ultimately, at asymptotically high  $Q^2$ , we connect  
 w/ perturbative QCD, and so the quarks will  
 receive a  $m_q \ln \frac{m_q^2}{Q^2}$  divergence which must be  
 regulated and renormalized

$$g^{\mu\nu} D_{\mu\nu}^{(1)} = 3$$

$$g^{\mu\nu} D_{\mu\nu}^{(2)} = (1 - \frac{v^2}{q^2})$$

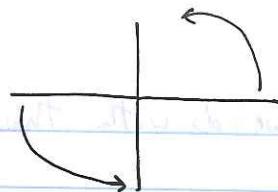
$$g^{\mu\nu} d_{\mu\nu}^{(1)} = 3$$

$$g^{\mu\nu} d_{\mu\nu}^{(2)} = 1 + 2 \frac{v^2}{q^2}$$

$$\delta M = \frac{i}{2M} \frac{\alpha_{f.s.}}{(2\pi)^3} \int_R d^4q \frac{1}{q^2 + i\epsilon} \left\{ \begin{array}{l} -3 T_1(v, q^2) + (1 - \frac{v^2}{q^2}) T_2(v, q^2) \\ 3 q^2 t_1(v, -q^2) - (1 + 2 \frac{v^2}{q^2}) q^2 t_2(v, -q^2) \end{array} \right.$$

197

Wick rotate  $v \rightarrow iv$   
 $(q^0 \rightarrow iq^0)$



$$\delta M = \frac{1}{2M} \frac{\alpha_{f.s.}}{(2\pi)^3} \int dv \int d^3 q \frac{1}{v^2 + \vec{q}^2} \begin{cases} -3T_1(iv, -q^2) + (1 - \frac{v^2}{v^2 + \vec{q}^2}) T_2(iv, -q^2) \\ -3(v^2 + \vec{q}^2) t_1(iv, -q^2) + (1 + 2\frac{v^2}{v^2 + \vec{q}^2})(v^2 + \vec{q}^2) t_2 \end{cases}$$

$$\int d^3 q = 4\pi \int_0^\infty dq q^2$$

variable transform

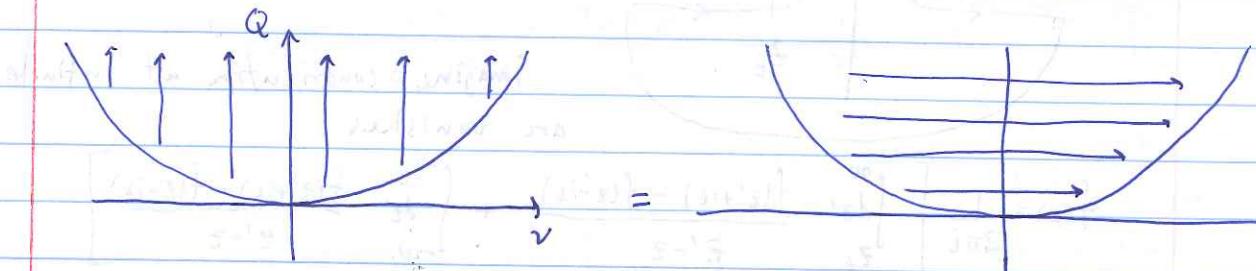
$$Q^2 = v^2 + \vec{q}^2$$

$$2QdQ = 2|q|dq, \quad |q| = \sqrt{Q^2 - v^2}$$

$$= \frac{1}{2M} \frac{\alpha_{f.s.}}{(2\pi)^3} \int_{-\infty}^{\infty} dv \frac{4\pi}{2} \int_v^\infty dQ \frac{2Q}{Q^2} \frac{\sqrt{Q^2 - v^2}}{Q^2} \begin{cases} -3T_1(iv, Q^2) + (1 - \frac{v^2}{Q^2}) T_2(iv, Q^2) \\ -3Q^2 t_1(iv, Q^2) + (1 + 2\frac{v^2}{Q^2}) Q^2 t_2(iv, Q^2) \end{cases}$$

These are correct

Consider the integration region



$$\int_{-\infty}^{\infty} dv \int_{v^2}^{\infty} dQ^2$$

$$\int_0^{\infty} dQ^2 \int_{-Q}^Q dv$$

$$\delta M = -\frac{\alpha_{f.s.}}{8M\pi^2} \int_0^{\infty} dQ^2 \int_{-Q}^Q dv \frac{\sqrt{Q^2 - v^2}}{Q^2} \begin{cases} 3T_1(iv, Q^2) - (1 - \frac{v^2}{Q^2}) T_2(iv, Q^2) \\ 3Q^2 t_1(iv, Q^2) - (1 + 2\frac{v^2}{Q^2}) Q^2 t_2(iv, Q^2) \end{cases}$$

198

Now what do we do with this?

We can use dispersion integrals to determine the functions  $T_i(iv, \alpha^1)$ ,  $t_i(iv, \alpha^2)$

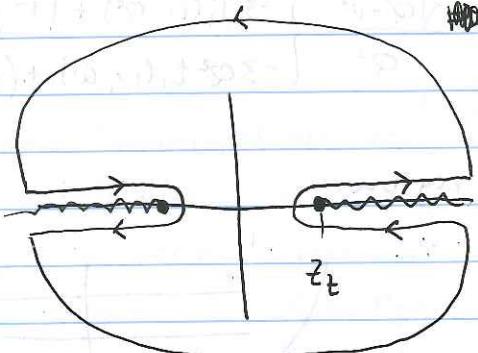
Recall: we can use the imaginary part of a function to determine the entire function

$$f(z) = \frac{1}{2\pi i} \oint dz' \frac{f(z')}{z' - z} \quad \text{Cauchy's Integral formula}$$

In our case, the function is crossing symmetric

$$f(-z) = f(z)$$

$$\Rightarrow f(z^*) = f^*(z) \quad \text{also}$$



we can deform contour

imagine contribution at infinite arc vanishes

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \left[ \int_{z_t}^{\infty} dz' \frac{f(z'+i\varepsilon) - f(z'-i\varepsilon)}{z' - z} + \int_{-\infty}^{-z_t} dz' \frac{f(z'+i\varepsilon) - f(z'-i\varepsilon)}{z' - z} \right] \\ &= \frac{1}{2\pi i} \left[ \int_{z_t}^{\infty} dz' \frac{f(-z''+i\varepsilon) - f(-z''-i\varepsilon)}{-z'' - z} \right] \end{aligned}$$

(crossing symmetry)  $f(-z''+i\varepsilon) = f(z''-i\varepsilon)$

$$= \frac{1}{2\pi i} \int_{z_t}^{\infty} dz' \left[ f(z'+i\varepsilon) - f(z'-i\varepsilon) \right] \left[ \frac{1}{z'-z} + \frac{1}{z'+z} \right]$$

$$= \frac{1}{2\pi i} \int_{z_t}^{\infty} dz' \left[ f(z'+i\varepsilon) - f^*(z'-i\varepsilon) \right] \frac{2z'}{z'^2 - z^2}$$

$$= \frac{1}{2\pi} \int_{z_t}^{\infty} dz' \frac{2z'}{z'^2 - z^2} 2\operatorname{Im} f(z'+i\varepsilon) \quad \text{just above cut}$$

Dispersion Integral

What if the contribution at the infinite arc does not vanish?

Can we suppress by a power-law?

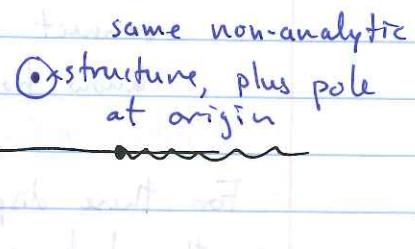
Consider

$$g(z) = \frac{f(z)}{z^2}$$

we first try  $\frac{1}{z^2}$  in our case because of the known crossing symmetry  $f(-z) = f(z)$

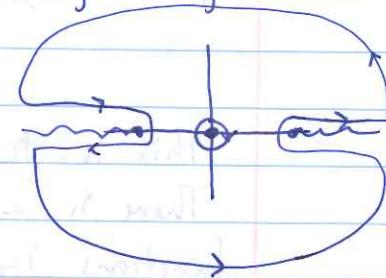
Now apply Cauchy Integral Again

$$g(z) = \frac{1}{2\pi i} \oint dz' \frac{f(z')}{z'^2(z'-z)}$$



Assuming the infinite arc now vanishes, going through the same procedure, we will get

$$g(z) = \frac{1}{2\pi} \int_{z_t}^{\infty} dz' \frac{2z'}{z'^2(z'^2 - z^2)} 2\text{Im } f(z'_+) - \frac{f(0)}{z^2}$$



sum of two contours

$$\Rightarrow f(z) = \frac{z^2}{2\pi} \int_{z_t}^{\infty} dz' \frac{2z'}{z'^2(z'^2 - z^2)} 2\text{Im } f(z'_+) + f(0)$$

cancels residue pole at origin

Subtracted dispersion relation

In our case, this single subtraction is sufficient.

The trouble is, we can measure experimentally  $\text{Im } f(z')$  but we can not measure  $f(0)$ . And since QCD is the underlying theory, we can not compute it either.